Moisil and many-valued logic. Personal recollections

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Abstract

Gr. Moisil worked in numerous areas, including mathematical logic. My work in the late 50's and early 60's dealt with many-valued logic, and I corresponded with Moisil about problems in this area. I also met him personally at two conferences in August 1962. Besides telling something about these events, this note contains a brief description about Moisil's contributions to many-valued logic, as well as those parts of my work in this area that were influenced by Moisil.

1 Introduction

I became interested in many-valued logic during my studies at the University of California, Berkeley, 1956-57. In a many-valued system of logic the principle *"every proposition is either true or false"* is not valid. Instead of two truth-values T (truth) and F (falsity), there are three or more truth-values. Further details, such as the interpretation of the additional truth-values, are fixed in each particular system of many-valued logic. Although there were many forerunners of many-valued logic (see, for instance, [11]), the actual discovery of the field is usually credited to the work of Łukasiewicz and Post about 1920.

Of course, everything boils down to the principle of every proposition being either true or false, also called the *law of the excluded middle.* Aristotle was certainly familiar with this law and, therefore, many-valued logics are often referred to as *non-Aristotelian.* However, he did not accept the law without reservations because it is not applicable to propositions referring to future contingent events. Chrysippus, a founder of the Stoic school, is usually considered to be the actual inventor of this law. This leads to the term *non-Chrysippian* logics, important in our discussions below.

Grigore Constantin Moisil published more than 40 papers about many-valued logics and related areas. He used the term non-Chrysippian, sometimes also *non-classical* logics. The oldest of his papers in this area date to early 40's (see [5]), while he published also much later the collections [7, 8]. Especially the collection [8] is very comprehensive.

I became first aware of Moisil and his work in many-valued logic through reviews published in *Mathematical Reviews.* The libraries, especially in Berkeley but also in Turku, had at least some of his Romanian papers. After writing to him, I got an answer which lead to a couple of more letters.

Already before the correspondence, I was impressed by the broad scope of Moisil's publications. He had worked in differential equations, for instance, first order partial differential equations of Vecua. He had papers in number theory, Galois theory, representation theory of abelian groups, as well as in singular Riemann spaces, especially geodetics in such spaces, theory of elasticity and electronic circuit theory. Later on I learned that his doctoral work with Volterra was some kind of a precursor of functional analysis.

My own main work in many-valued logic dealt with *functional completeness and Sheffer functions,* [10, 12, 13, 14]. (See also [3] for interconnections to other areas.) With Moisil I discussed matters considered in [11]: history and generalizations of truth-functions to the many-valued case.

Moisil drew my attention to the work of N.A. Vasiliev published around 1910 but fairly unknown. The work comes close to a modern conception of a many-valued logic. Vasiliev has in his logic *three forms of judgment*: simple affirmation *S is P*, simple negation *S is non-P*, and the combination of the two (indifferent judgment) *S is simultaneously P and non-P*. The law of the excluded fourth is valid. Vasiliev constructs a consistent system on the basis of these suppositions. It is emphasized that the theory is directed against conceiving the law of the excluded middle in too general a fashion.

Moisil's approach was very broad, often algebraic, [6]. He was also interested in interconnections between many-valued and intuitionistic logic, as well as in Gödel's "Königsberg papers", for instance, [2].

Some of the aspects discussed with Moisil concerning generalizations of truth-functions will be presented below in Section 4. The next section, Section 2, deals with two conferences that took place in August 1962. These were the only occasions I met Moisil personally. Both in our correspondence and personal meetings Moisil was very friendly and encouraging. I did not have any long discussion with him, he was very busy meeting with other people. This was very understandable because almost all famous logicians were present. Certainly there was never such a concentration of great logicians in Finland as in August 1962.

Section 3 is based mainly on [4] and deals some of the work of Moisil on many-valued logics.

2 Conferences in August 1962

The International Congress of Mathematicians, the big ICM, took place in Stockholm in August 1962. As customary, there were plenary lectures, invited lectures in various sections, and brief presentations in sections. Logic and foundations of mathematics were dealt with in the congress section 1. All of the following remarks deal with this section.

Alonzo Church gave a plenary lecture, and Dana Scott an invited one. Besides Moisil and myself, brief presentations were given, for instance, by Boone, Davis, van Dalen, Ginsburg, Kalmar, Maltšev, Nerode, Sacks, Schoenfield, Skolem, Smullyan, Tarski and Turquette.

There was no refereeing for the brief presentations, and the time allowed for everybody was only some fifteen minutes. Still quite a group of famous people gave presentations! This, I think, indicates that there were very few conferences those days. Another indication of this state of affairs is that all the famous people came to the satellite conference in Helsinki, even if they had no formal duties there.

The inevitable conclusion is that there were too few conferences at the time when Moisil was active, whereas nowadays there are perhaps too many.

The talks by me and Moisil were in different sessions of section 1. I attended all sessions, so I was present in Moisil's talk. But I do not remember whether he was present in my talk; the only questions posed to me came from R.L. Goodstein. There is no record of Moisil's talk in the booklet of brief presentations but the title was "La logique à trois valeurs et ses applications".

The chairmen of the sessions were famous people. My chairman was Abraham Frænkel, Moisil's S.C. Kleene. The latter was very formal in his wordings. He just announced the person and the title of the talk. After the talk he would only say: "Mr. Moisil's paper is open for discussion." As far as I remember, there was no discussion. I was certainly too shy to initiate any. It was a nice interplay between languages: Kleene made the announcements in English, and Moisil gave his talk in French. There was no interpretation. But some of the Russian talks were translated into English. I still remember one correction Tarski made to a translation. When the translator said "a bounded number of quantifiers", Tarski corrected "a bounded number of changes of quantifiers".

The second conference where I met Moisil was the Colloquium on Modal and Many-Valued Logics held in Helsinki on 23–26 August, 1962. It took place under the auspices of the Division of Logic, Methodology and Philosophy of Science of the International Union of History and Philosophy of Science (DLMPS/IUHPS, there were jokes about the length of this abbreviation!). The timing made it one of the satellite conferences of the ICM. The originally planned topic of the Colloquium was "non-classical logics" but, later on, the more specific topic was agreed upon. The main organizer of the Colloquium was G.H. von Wright.

The whole logic community in the ICM , more or less, came from Stockholm to Helsinki, and there were also many additional logicians in Helsinki because of the DLMPS/IUHPS activities. From the subsequent lists of names one can see what an impressive crowd was present. All lectures in the Colloquium were invited ones, there were no submitted papers.

The lecturers in the following list are given in the alphabetic order:

A.R. Anderson (Yale University), C.C. Chang (UCLA), P.T. Geach (University of Birmingham), S. Halldén (University of Uppsala), K.J. Hintikka (University of Helsinki), H. Hiż (University of Pennsylvania), S. Kanger (University of Stockholm), S.A. Kripke (Harvard University), E.J. Lemmon (University of Oxford), Ruth Barcan Marcus (Roosevelt University, Chicago), Gr.C. Moisil (University of Bucharest), R. Montague (UCLA), A. Mostowski (University of Warsaw), J. Porte (CNRS, Paris), A.N. Prior (University of Manchester), H. Rasiowa (University of Warsaw), N. Rescher (University of Pittsburgh), A. Salomaa (University of Turku), T.J. Smiley (University of Cambridge), E. Stenius (Åbo Akademi), A.R. Turquette (University of Illinois), and L. Åqvist (University of Uppsala).

Many sessions of the Colloquium were also attended by the majority of the delegates to the General Assembly of the DLMPS/IUHPS, for instance, Alonzo Church and S.C. Kleene. Moreover, at least the following delegates to the General Assembly served as chairmen of various sessions of the Colloquium: K. Adjukiewicz (University of Warsaw), H.B. Curry Pennsylvania State University), J.B. Rosser (Cornell University), and A. Tarski (University of California, Berkeley).

The weather was not very good. Some people, notably R. Montague, had their overcoats on at all times, also inside. There was a conference dinner, at least Church and Curry were in the same table with me. There was no conference excursion.

Moisil had his talk in the same session as me. His talk was before mine. Mostowski was the chairman. I was nervous because of my talk, and did not really listen to Moisil's talk but studied the matters only later on. Those days the only possibility was to use chalk and blackboard, so you had to remember your stuff. Mostowski was very considerate and apparently noticed that I was nervous. He did not ask his very interesting and difficult question about decidability during the actual session but only afterwards.

Moisil's and my own talks were published later on, [4, 14]. My presentation in Section 3 is very much based on [4]. The matters in [14] will not be dealt with in this article because they are not among the topics I discussed with Moisil.

3 Moisil: les logiques non-Chrysippiennes

Moisil's contribution [4] for the Helsinki satellite conference is an overview of some of his work on many-valued logic. It reflects very well his tendency to combine the axiomatic approach to the truth-table approach, as well as his interest in lattice theory. In this section we mention some ideas from this paper. We use the same notation here as in the last section. In an n-valued logic the truth values are $1, \ldots, n$, with 1 being the highest. Truth-functions $f(x, y)$ are given by tables: the value $f(x, y)$ is in the intersection of the x-th row and y-th column.

Moisil considers the truth-tables of the three-valued logic of Łukasiewicz. The negation N interchanges 1 and 3 but keeps the value 2 fixed: $N(2) = 2$. The conjunction K, disjunction A and implication C have the truth-tables

$$\begin{array}{|ccc} \hline 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{array}\ , \qquad \begin{array}{|ccc} \hline 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{array}\ , \qquad \begin{array}{|ccc} \hline 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{array}\ .$$

The axiomatic version of the logic uses two rules of inference: substitution and *modus ponens*. Moisil considers the following axioms:

$$CpCqp,\ CCpqCCqrCpr,\ CCNpNqCqp,\ CCCpNppp.$$

Somewhat differently from Moisil's presentation, we have used here the customary parenthesis-free notation.

Moisil points out that, using the truth-tables above, one can get rid of the disjunction and conjunction by defining Apq as $CCpqq$, and Kpq as $NANpNq$. He then investigates the distributive lattice structure of the logic obtained.

Moisil also investigates the modal aspects of the three-valued logic. He considers four modalities: possibility, necessity, impossibility and non-necessity. We restrict the attention to the first two. Following Moisil, we denote possibility by μ and necessity by ν.

By definition, for the value sequence $1, 2, 3$, the function μ assumes the value sequence $1, 1, 3$, and the function ν the value sequence $1, 3, 3$. Thus, the intermediate value is completely avoided. The definitions are also in accordance with the Aristotelian principle: *Ab esse ad posse valet consequentia.*

Possibility can also be defined in terms of negation and implication by letting μp to be $CNpp$.

Moisil makes the general comment that an axiomatization should yield an algebraic system with easy operation rules. He does not comment upon to what extent this goal is achieved in the systems he investigates, except that the Łukasiewicz 3-valued logic is particularly strong in this respect. (See also [6].)

Moisil investigates the notion of *compatibility*: when are two propositions p and q compatible? A false proposition is not compatible with any other, whereas two "sufficiently true" ones are compatible among themselves. The considerations lead Moisil to a 4-valued logic, where compatibility is defined by the truth-table

$$\begin{array}{|cccc} \hline 1 & 1 & 1 & 4 \\ 1 & 1 & 4 & 4 \\ 1 & 4 & 1 & 4 \\ 4 & 4 & 4 & 4 \end{array}\ .$$

Thus, the intermediate truth-values do not appear in the table. Moisil's idea was that two "equally true" propositions are compatible (if they are not false). This is the case even for the truth-value 3.

When considering the validity of certain specific propositions, Moisil introduces the following two modifications of the Łukasiewicz implication:

$$\begin{array}{|ccc} \hline 1 & 2 & 3 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{array}\,, \qquad \begin{array}{|ccc} \hline 1 & 3 & 3 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{array}\,.$$

Thus, the differences concern the positions $(1,2),(2,2),(2,3)$.

Moisil was also interested (especially in Section VI of [4]) in generating functions in terms of other functions. This was a topic of my main interest. I will mention one example, not from Moisil's paper.

Consider the three-valued logic, $n = 3$. There are altogether 27 possible truth-functions of one variable. How to generate them in terms of some basic functions? At least three functions are needed for this purpose. We define three functions $\{a, b, c\}$ by their value sequences 231, 132 and 223, respectively. This means that, for instance,

$$a(1) = 2,\ a(2) = 3,\ a(3) = 1.$$

Thus, a is the circular permutation (123), b is the transposition (23), whereas g assumes only 2 values and maps 1 to 2 but keeps 2 and 3 fixed.

The following array lists all of the 27 functions, giving in each case the value sequence and a shortest possible composition sequence.

name	values	composition		name	values	composition
1	111	ca^2ca^2	•	2	112	ca^2
3	113	cba	•	4	121	aca^2
5	122	a^2cab	•	6	123	b^2
7	131	$acba$	•	8	132	b
9	133	a^2ca	•	10	211	a^2ca^2
11	212	$acab$	•	12	213	ba
13	221	cab	•	14	222	ca^2c
15	223	c	•	16	231	a
17	232	ac	•	18	233	a^2cb
19	311	$abcba$	•	20	312	a^2
21	313	aca	•	22	321	ab
23	322	a^2c	•	24	323	acb
25	331	ca	•	26	332	cb
27	333	ca^2ca	•			

Thus, altogether 10 different functions are represented by words of length ≤ 2. Additionally, 6 functions are represented by words of length 3, and 6 further functions by words of length 4. The remaining exceptional functions 1,5,10,19,27 require a longer word for their representation. The length considerations lead to matters discussed in [3]. Observe that [1] is a very early paper along these lines.

4 Many-valued truth functions

This section describes some of my work concerning generalizations of the connectives in two-valued propositional calculus into many-valued logic. The approach is more truth-functional than axiomatic. This is an area where I was influenced by discussions with Moisil. I will restrict here the attention to the generalizations of the material implication. More details, as well as generalizations of other connectives, are contained in [11].

Binary functions $f(x, y)$ over a finite set (here the set of truth-values) are conveniently defined by tables, where the values of x are read from the rows and those of y from the columns. We already considered the three-valued Łukasiewicz implication. It is generalized to the n-valued case. Then its truth-function is defined by $c(x, y) = \max(1, 1 - x + y)$. For $n = 6$, its truth-table is

1	2	3	4	5	6
1	1	2	3	4	5
1	1	1	2	3	4
1	1	1	1	2	3
1	1	1	1	1	2
1	1	1	1	1	1

(We have omitted here the indices $1, \ldots, 6$ from the rows and columns. The ordering of the truth-values is the same as in Section 3.)

Before we continue, some clarifying remarks are in order. We always use the numbers $1, \ldots, n$ to denote truth-values, 1 being the "greatest". The numbers $1, \ldots, d,\ d < n$ refer to *designated* truth-values. (Assertible propositions should get only designated truth-values. This is a notion standard in many-valued logic, see [9].) Arithmetical operations are carried out modulo n.

Łukasiewicz did not motivate his choice of the particular implication function presented above. However, his function satisfies most of the requirements for a generalized implication listed below. This is the case only when the number of designated truth-values is one.

The following list, [11], of conditions for a generalized implication $c(i, j)$ is based on the literature and discussions with some people, notably Moisil. Intuitive explanations are added to some conditions. The range of i, j and h is the set of truth-values.

1. For all $i \leq d$ and $j > d,\ c(i, j) > d$. (Modus ponens is valid.)
2. For some i and j, $c(i, j) \leq d$ and $c(j, i) > d$. (Implication is not commutative.)
3. For some $j > d,\ c(i, j) \leq d$. (The minor premise is not superfluous in modus ponens.)
4. For some $i, j \leq d,\ c(i, j) \leq d$.
5. For all $i,\ c(i, i) \leq d$.
6. For all i, j, if $i \geq j$, then $c(i, j) \leq d$.

7. For all i, j, if $i < j$ then $c(i, j) > d$.

8. For all i, j, h, if $i > j$ then $c(h, i) \geq c(h, j)$.

9. For all i, j, h, if $i > j$ then $c(i, h) \leq c(j, h)$.

10. Whenever $c(h, i) \leq d$ and $c(i, j) \leq d$, then also $c(h, j) \leq d$. (Transitivity of implication.)

11. $c(i, j) > d$ if and only if both $i \leq d$ and $j > d$.

At least these conditions, and perhaps some others, should be considered when one wants to study the question which among the functions in n-valued propositional calculus might plausibly be called implication functions. There are mutual dependencies among the conditions. For instance, condition 11 alone implies conditions 1–6 and 10. Condition 5 implies both 3 and 4, and conditions 6 and 7 together imply 1–5 and 10. Moreover, 11 is not consistent with 7 in some cases. By grouping together some of the conditions, one obtains quite satisfactory results. Consider the following sets of conditions.

- C_1 consists of conditions 1–4.
- C_2 consists of 1,2,5.
- C_3 consists of 1,2,5,10.
- C_4 consists of 6,7.
- C_5 consists of 6–9.
- C_6 consists of 11 alone.
- C_7 consists of 8,9,11.

Each of the sets C_i is strong enough to determine implication uniquely in the ordinary two-valued case where $n = 2,\ d = 1$. The sets $C_1 - C_5$ form a sequence where each set contains stronger conditions than the preceding one, that is, a function satisfying C_i also satisfies C_{i-1}. The same holds true with respect to the sequence $C_1 - C_3, C_6, C_7$. Thus, the sets $C_1 - C_7$ give a pretty good view of conditions of various strengths, as well as the contradictoriness of conditions 7 and 11. Moreover, each of the sets $C_1 - C_7$

is *consistent*: for all d, n, there is a function satisfying every condition in the set. Each of the sets $C_1 - C_6$ consists of *strongly independent* and the set C_7 of *weakly independent* conditions. The difference depends on whether we can find examples (separating any two conditions in the set as regards satisfaction) for all d, n or for some d, n.

The Łukasiewicz implication considered above satisfies conditions 1–10 and 12, provided $d = 1$. If $d > 1$ then it satisfies none of the conditions 1, 7 or 10. The condition 11 is never satisfied by the Łukasiewicz implication.

Moisil emphasized the interconnection between the axiomatic approach and the truth-table approach. A problem along these lines is to study the validity of two-valued tautologies under functions satisfying a certain set C_i of conditions. (We restrict the attention here to conditions for implication.) Such a study was carried out in [11] for tautologies considered as most important in divisions 2-5 of *Principia Mathematica.* If the implication satisfies the set C_7, then any tautology of the two-valued logic is valid. This follows because ordinary truth-table technique can be applied, with "true" and "false" replaced by "designated" and "undesignated".

The situation is quite different if one considers the other "strong" set C_5. Then in some cases things become very tricky. We mention one example.

Consider the "commutative principle"

$$\mathcal{CP}: \ (p \to (q \to r)) \to (q \to (p \to r))$$

and the set C_5. Assume that $n \geq 3,\ 1 \leq d < n$. A rather surprising result is that $\mathcal{CP}$ is completely independent of C_5. For all n, d, there is an implication function satisfying C_5 and making $\mathcal{CP}$ assertible, as well as another implication function satisfying C_5 and making it non-assertible.

In most cases the required functions are fairly easy to construct. The difficult case is where $d > 1$ and we are looking for a function $c(x, y)$ making $\mathcal{CP}$ assertible. Such a function can be defined as follows. We define first the values of the function when one of the arguments is either 1 or n. Then

$$c(n, i) = c(i, 1) = 1, \text{ for all } i,$$

and

$$c(1, i) = n \text{ if } i > 1;\ c(i, n) = n \text{ if } i < n.$$

The definition is completed by the following two equations (now both x, y differ from $1, n$):

$$c(x, y) = \min(n, d + \max(|x - d|, |y - d|)) \text{ if } x < y,$$

and

$$c(x, y) = \max(2, d - \max(|x - d|, |y - d|)) \text{ if } x \geq y.$$

As an example, we give the truth-tables of $c(x, y)$ when $n = 7$ and $d = 2, 3, 4, 5, 6$, successively:

$$\begin{array}{|ccccccc} \hline 1 & 7 & 7 & 7 & 7 & 7 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 2 & 4 & 5 & 6 & 7 \\ 1 & 2 & 2 & 2 & 5 & 6 & 7 \\ 1 & 2 & 2 & 2 & 2 & 6 & 7 \\ 1 & 2 & 2 & 2 & 2 & 2 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}, \qquad \begin{array}{|ccccccc} \hline 1 & 7 & 7 & 7 & 7 & 7 & 7 \\ 1 & 2 & 4 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 2 & 2 & 5 & 6 & 7 \\ 1 & 2 & 2 & 2 & 2 & 6 & 7 \\ 1 & 2 & 2 & 2 & 2 & 2 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array},$$

$$\begin{array}{|ccccccc} \hline 1 & 7 & 7 & 7 & 7 & 7 & 7 \\ 1 & 2 & 6 & 6 & 6 & 6 & 7 \\ 1 & 2 & 3 & 5 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 3 & 3 & 6 & 7 \\ 1 & 2 & 2 & 2 & 2 & 2 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}, \qquad \begin{array}{|ccccccc} \hline 1 & 7 & 7 & 7 & 7 & 7 & 7 \\ 1 & 2 & 7 & 7 & 7 & 7 & 7 \\ 1 & 2 & 3 & 7 & 7 & 7 & 7 \\ 1 & 2 & 3 & 4 & 6 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 4 & 4 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array},$$

$$\begin{array}{|ccccccc} \hline 1 & 7 & 7 & 7 & 7 & 7 & 7 \\ 1 & 2 & 7 & 7 & 7 & 7 & 7 \\ 1 & 2 & 3 & 7 & 7 & 7 & 7 \\ 1 & 2 & 3 & 4 & 7 & 7 & 7 \\ 1 & 2 & 3 & 4 & 5 & 7 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}.$$

It is easy to see that $c(x, y)$ satisfies the conditions in C_5. (For instance, each row should be non-decreasing and each column non-increasing.) It is much more difficult to show that $c(x, y)$ makes $\mathcal{CP}$ assertible. Then ordinary truth-table technique works if one of the variables assumes the truth-value 1 or n. Otherwise, one applies induction on i and shows that $\mathcal{CP}$ assumes a designated value always when the variables belong to the intersection of the two closed intervals $(d - i, d + i)$ and $(2, n - 1)$.

In general, the truth-table technique has to be modified when one has to demonstrate that some well-formed formula is assertible in the n-valued propositional calculus, without specifying the number n. Then one cannot simply list all combinations of values for the variables. One has to consider some systems of truth-tables or use inductive procedures.

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